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Working Paper No. 10/2006

May 2006

APPLIED MATHEMATICS

WORKING PAPER SERIES

A Multivariate Time-Changed Lévy Model for Financial Applications

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Abstract

The purpose of this paper is to define a bivariate Lévy process by subordination of a Brownian motion. In particular we investigate a generalization of the bivariate Variance Gamma process proposed in Luciano and Schoutens [8] as a price process. Our main contribution here is to introduce a bivariate subordinator with correlated Gamma margins. We characterize the process and study its dependence structure. At the end we also propose an exponential Lévy price model based on our process.

Mathematics Subject Classification (2000): 60G51, 60E07.

Keywords: Lévy processes, multivariate subordinators, dependence, multivariate asset modelling.

Introduction

The purpose of this paper is to introduce a bivariate Lévy process constructed by subordination of a Brownian motion with independent components by a bivariate subordinator. Subordinate process are widely studied both theoretically and in finance applications. In particular the class of Variance Gamma processes is extensively studied in finance. They have been first introduced in literature by Madan and Seneta [9], [10] as models for stocks return. Let $\tilde{B}$ be a Brownian motion and G a Gamma process and let them be independent. The Variance Gamma process $X = \{X(t) : t \geq 0\}$ is defined by subordination as: $X(t) = \tilde{B}(G(t))$, $\forall t \geq 0$. In [9], [10] they considered a symmetric model (in which the Brownian motion has zero drift), while they studied the general case in [11]. A first generalization to multivariate models is proposed by Madan and Seneta [10] themselves. Actually, the model we present here is inspired by the bivariate generalization of the Variance Gamma made by Luciano and Schoutens [8]. They considered two independent Brownian motion subordinated by a common Gamma process. Also Cont and Tankov [5] proposed to subordinate a bivariate Brownian motion with a common Gamma subordinator.

The main contribute here is to introduce a bivariate subordinator $\boldsymbol{G}$ with correlated Gamma margins. We define it starting from a bivariate infinitely divisible law which is a generalization of the McKay Gamma distribution [1]. We define the time-changed process $\boldsymbol{Y}$ by

$$\boldsymbol{Y}(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = \begin{pmatrix} \tilde{B}_1(G_1(t)) \\ \tilde{B}_2(G_2(t)) \end{pmatrix}. \tag{0.1}$$

We prove that the process is a pure jump with correlated Variance Gamma margins. We also prove that the model investigated in [8] can be derived from $\boldsymbol{Y}$ with a particular choice of the parameters.

The process $\boldsymbol{Y}$ can be generalized to a multivariate one, as we will underline at the end of its construction. The main difference consists in the number of parameters to handle; it is the reason why we investigate the two dimensional case.

The paper is organized as follows. After some preliminary notion in Section 1, Section 2 is entirely devoted to introducing the process and characterize it as a Lévy one. The process that inspired our generalization arise from a financial setting and was proposed as a price model. In this model dependence between margins plays a central role. For this reason in Section 3 we analyze the dependence structure of $\boldsymbol{Y}$. Section 4 is devoted

to defining a price model starting from $\boldsymbol{Y}$ and to comparing our model with the ones belonging to the Variance Gamma class. Finally to have a first idea of the behavior of the process both jointly and marginally we make some simulations in Section 5.

1 Preliminaries

We recall here the definition of Lévy process and the main related properties we will use throughout the paper; for a complete overview about this matter see [5] and [13].

Definition 1.1. *A cadlág stochastic process* $\boldsymbol{X} = \{\boldsymbol{X}(t), t \geq 0\}$ *on a probability space* $(\Omega, \mathcal{F}, P)$ *with values in* $\mathbb{R}^d$ *such that* $\boldsymbol{X}(0) = 0$ *is called a Lévy process if it has independent and stationary increments and it is stochastically continuous, i.e.* $\forall \varepsilon > 0,\ \lim_{h \to 0} P(|\boldsymbol{X}(t+h) - \boldsymbol{X}(t)| \geq \varepsilon) = 0$.

Let $\boldsymbol{X}(t)$ be a Lévy process, it can be proved that for any t the random vector $\boldsymbol{X}(t)$ has an infinitely divisible distribution and conversely if F is an infinitely divisible distribution then there exists a Lévy process $\{\boldsymbol{X}(t), t \geq 0\}$ such that the distribution of $\boldsymbol{X}(1)$ is F, moreover if $\boldsymbol{X}(t)$ and $\boldsymbol{X}'(t)$ are Lévy processes in law on $\mathbb{R}^d$ such that $\boldsymbol{X}(1)$ and $\boldsymbol{X}'(1)$ have the same distributions then $\boldsymbol{X}(t)$ and $\boldsymbol{X}'(t)$ are identical in law (see [13], Theorem 7.10). We will use this property to define a new Lévy process. The characteristic function of a Lévy process is fundamental in its construction. It admits the following *Lévy-Khinchin representation*:

$$\psi_{\boldsymbol{X}(t)}(z) = E[e^{i\langle \boldsymbol{z}, \boldsymbol{X}(t)\rangle}] = e^{t\Psi_{\boldsymbol{X}}(\boldsymbol{z})}, \quad \boldsymbol{z} \in \mathbb{R}^d,$$

with

$$\Psi_{\boldsymbol{X}}(\boldsymbol{z}) = -\frac{1}{2}\langle \boldsymbol{z}, A\boldsymbol{z}\rangle + i\langle \gamma, \boldsymbol{z}\rangle + \int_{\mathbb{R}^d} (e^{i\langle \boldsymbol{z}, \boldsymbol{x}\rangle} - 1 - i\langle \boldsymbol{z}, \boldsymbol{x}\rangle 1_{|\boldsymbol{x}|\leq 1)})\nu(d\boldsymbol{x}),$$

where A is a symmetric $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a positive random measure on $\mathbb{R}^d$. Observe that for each $\boldsymbol{z} \in \mathbb{R}^d$, $\psi_{\boldsymbol{X}(t)}(z) = (\psi_{\boldsymbol{X}(1)}(z))^t$. Moreover given (γ, A, ν) the corresponding Lévy process is unique in distribution, (γ, A, ν) is called the Lévy triplet of the process. $\Psi_{\boldsymbol{X}}(z)$ is named characteristic exponent of $\boldsymbol{X}$. If $\{X(t), t \geq 0\}$ is a real Lévy process of finite variation with Lévy triplet $(\gamma, 0, \nu)$ its characteristic exponent can be expressed as (see [5]):

$$\Psi_X(z) = im \cdot z + \int_{\mathbb{R}} (e^{iz \cdot x} - 1)\nu_X(dx), \tag{1.1}$$

where $m = \gamma - \int_{|x|\leq 1} x\nu(dx)$.

In the next section we focus our attention on a particular class of Lévy process, the subordinators, that are real increasing Lévy processes. They have no diffusion component and are of finite variation, thus (1.1) holds. For a characterization see [5] (Proposition 3.10). More precisely we are interested in the multivariate generalization of subordinators. A *multivariate subordinator* is a Lévy process on $\mathbb{R}^d_+ = [0,\infty)^d$. The trajectories of multivariate subordinators are increasing in each coordinate. See Barndoff-Nielsen et al. [2] for the main properties of such processes. The characteristic exponent of a multivariate subordinator has the same expression of (1.1), as it is proved in [2], Proposition 3.1:

$$\Psi_{\boldsymbol{X}}(\boldsymbol{z}) = i\langle \boldsymbol{m}, \boldsymbol{z}\rangle + \int_{\mathbb{R}^d} (e^{i\langle \boldsymbol{z},\boldsymbol{x}\rangle} - 1)\nu(d\boldsymbol{x}), \tag{1.2}$$

where $\boldsymbol{m} \in \mathbb{R}^d_+$ and $\nu_{\boldsymbol{X}}$ is the Lévy measure of $\boldsymbol{X}$.

The starting point of our model is the Variance Gamma process. There are different way to define this process, we adopt here the definition more suitable to our aim.

A *Variance Gamma* process is a real Lévy process $X_{VG} = \{X_{VG}(t), t \geq 0\}$ obtained as a Brownian motion with drift time-changed by a Gamma process.

A process $\{G(t), t \geq 0\}$ is a Gamma process with parameters (a, b) if it is a Lévy process so that the defining distribution of $X(1)$ is Gamma with parameters (a, b) (shortly $\mathcal{L}(X(1)) = \Gamma(a, b)$). It is a finite variation Lévy process, therefore its characteristic function follows the representation (1.1). An easy calculation shows that $m = 0$. Its Lévy triplet is

$$\gamma = \tfrac{a(1-exp(-b))}{b},$$
$$A = 0$$
$$\nu(dx) = a\exp(-bx)x^{-1}\mathbf{1}_{(0,+\infty)}(x)dx.$$

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $\{G(t), t \geq 0)$ be a Gamma process with parameters $(\frac{1}{\nu}, \frac{1}{\nu})$ and $\sigma > 0$, θ be real parameters, then the process X_{VG} is defined as

$$X_{VG}(t) = \theta G(t) + \sigma B(G(t)).$$

The characteristic function of X_{VG} is the following,

$$\psi_{X_{VG}(t)}(u) = [\psi_{X_{VG}(1)}(u)]^t = (1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-\frac{t}{\nu}}.$$

Madan et al. (1988) [11] showed that the Variance Gamma process may be expressed as the difference of two Gamma processes. This characterizations allows to determine the

Lévy measure

$$\nu_{VG}(dx) = \begin{cases} Cexp(Nx)|x|^{-1}dx, & x < 0 \\ Cexp(-Mx)x^{-1}dx & x > 0, \end{cases}$$

where

$$C = \tfrac{1}{\nu} > 0$$
$$N = (\sqrt{\tfrac{1}{4}\theta^2\nu^2 + \tfrac{1}{2}\sigma^2\nu} - \tfrac{1}{2}\theta\nu)^{-1} > 0$$
$$M = (\sqrt{\tfrac{1}{4}\theta^2\nu^2 + \tfrac{1}{2}\sigma^2\nu} + \tfrac{1}{2}\theta\nu)^{-1} > 0$$

The Lévy measure has infinite mass, therefore the process has infinitely many jumps in any finite time interval. Since

$$\int_{-1}^{1} |x|\nu_{VG}(dx) < \infty,$$

the process' paths are of finite variation. The Variance Gamma process has no Brownian component and its Lévy triplet is given by $(\gamma, 0, \nu_{VG}(dx))$, where

$$\gamma = \frac{-C(N(exp(-M) - 1) - M(exp(-N) - 1))}{MN}.$$

We end this section with a bivariate version of a theorem that plays a central rule in the characterization in terms of Lévy triplet of the process we are going to construct. The univariate version is Theorem 30.1 in Sato [13], while the general version and its proof are in [2] (Theorem 3.3). The statement and the proof of the theorem require the introduction of the multi-parameter process (the definition can be stated in a more general setting: for a complete treatment see again [2]). Consider two independent Lévy processes $X_1(t)$, $X_2(t)$. The stacked process $\boldsymbol{X}(t) = (X_1(t), X_2(t))$ is a then a Lévy process on $\mathbb{R}^2$. Consider the multi-parameter $\boldsymbol{s} = (s_1, s_2) \in \mathbb{R}^2_+$ and the partial order on $\mathbb{R}^2_+$:

$$\boldsymbol{s}^1 \preceq \boldsymbol{s}^2 \;\Leftrightarrow\; s_1^1 \leq s_1^2,\;\; s_2^1 \leq s_2^2.$$

Define the multi-parameter process $\{\boldsymbol{X}(\boldsymbol{s}),\, \boldsymbol{s} \in \mathbb{R}^2_+\}$ by

$$\boldsymbol{X}(\boldsymbol{s}) = (X_1(s_1), X_2(s_2)).$$

Theorem 1.1. *Let $\boldsymbol{G}$ be a bivariate subordinator with triplet $(\gamma_{\boldsymbol{G}}, 0, \nu_{\boldsymbol{G}})$ and let $\lambda_t = \mathcal{L}(\boldsymbol{G}(t))$. Let $\boldsymbol{X}(t)$ be a bivariate Lévy process with independent components and triplet $(\gamma_{\boldsymbol{X}}, \Sigma_{\boldsymbol{X}}, \nu_{\boldsymbol{X}})$ and let $\rho_{\boldsymbol{s}} = \mathcal{L}((X_1(s_1), X_2(s_2)))$. Define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t \geq 0\}$ by the following*

$$\boldsymbol{Y}(t) = (X_1(G_1(t)), X_2(G_2(t))),\;\; t \geq 0$$

then the process $\boldsymbol{Y}$ *is a Lévy process and*

$$E[e^{i\langle \boldsymbol{z}, \boldsymbol{Y}(t)\rangle}] = exp(t\Psi_{\boldsymbol{G}}(\log \psi_{\boldsymbol{X}}(\boldsymbol{z}))),\ \boldsymbol{z} \in \mathbb{R}^2_+,$$

where for any $\boldsymbol{w} = (w_1, w_2) \in \mathbb{C}^2$ *with* $Re(w_j) \leq 0,\ j = 1, 2$, *we let*

$$\Psi_{\boldsymbol{G}}(\boldsymbol{w}) = \langle \boldsymbol{m} \cdot \boldsymbol{w}\rangle + \int_{\mathbb{R}^2} (e^{\langle \boldsymbol{w}, \boldsymbol{x}\rangle} - 1)\nu(d\boldsymbol{x}).$$

Moreover the characteristic triplet $(\gamma_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ *of* $\boldsymbol{Y}$ *is as follows*

$$\begin{gathered}\gamma_{\boldsymbol{Y}} = \textstyle\int_{\mathbb{R}^2_+} \nu_{\boldsymbol{G}}(ds) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\rho_{\boldsymbol{s}}(d\boldsymbol{x}) + \langle \boldsymbol{m}, \gamma_{\boldsymbol{X}}\rangle, \\ \Sigma_{\boldsymbol{Y}} = diag(\boldsymbol{m} \cdot \Sigma_{\boldsymbol{X}}) \\ \nu_{\boldsymbol{Y}}(\boldsymbol{B}) = \nu_1(\boldsymbol{B}) + \nu_2(\boldsymbol{B})\end{gathered}$$

where ν_1 *and* ν_2 *are defined by* $\nu_1(\boldsymbol{0}) = 0$, $\nu_2(\boldsymbol{0}) = 0$ *and - for* $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^2 \setminus \boldsymbol{0})$ *- by*

$$\begin{gathered}\nu_1(\boldsymbol{B}) = \textstyle\int_{\mathbb{R}^2_+} \rho_{\boldsymbol{s}}(\boldsymbol{B})\nu_{\boldsymbol{G}}(d\boldsymbol{s}), \\ \nu_2(\boldsymbol{B}) = \textstyle\int_{\boldsymbol{B}} m_1 1_{A_1}(x)\nu_{X_1}(dx) + m_2 1_{A_2}(x)\nu_{X_2}(dx),\end{gathered}$$

where $x \in \mathbb{R}$, $\nu_{X_i},\ i = 1, 2$ *are the Lévy measures of the independent marginal processes of* $\boldsymbol{X}$ *and finally* $A_i = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : x_k = 0$ *for* $k \neq i,\ k = 1, 2\}$, $i = 1, 2$. *Finally if* $\boldsymbol{m} = \boldsymbol{0}$ *and* $\int_{|\boldsymbol{s}|\leq 1} |\boldsymbol{s}|^{\frac{1}{2}} \nu_{\boldsymbol{G}}(d\boldsymbol{s}) < \infty$, *then* $\Sigma = \boldsymbol{0}$, $\int_{|\boldsymbol{x}|\leq 1} |\boldsymbol{x}|\nu(d\boldsymbol{x}) < \infty$ *and* $\boldsymbol{Y}$ *has zero drift and is of bounded variation on any time interval almost surely.*

2 The $\alpha\beta$-Variance Gamma model

The construction of the Lévy process by subordination requires different steps.

2.1 Step 1: a bivariate subordinator

Lévy processes can be easily defined starting from infinitely divisible distributions. We begin hence with constructing a bivariate infinitely divisible distribution with correlated margins.

Let X_1, X_2 and Z be independent and infinitely divisible random variables, with characteristics functions respectively ψ_1, ψ_2 and ψ_Z. Define the random vector $\boldsymbol{W}$ as

$$\boldsymbol{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} X_1 + \alpha Z \\ X_2 + \beta Z \end{pmatrix}, \tag{2.1}$$

where α and β are two real parameters.

The bivariate distribution defined is infinitely divisible, as we can easily verify by the characteristic function $\psi_{\boldsymbol{W}}$ of $\boldsymbol{W}$:

$$\begin{aligned}\psi_{\boldsymbol{W}}(u_1,u_2) &= E[exp\{iu_1W_1 + iu_2W_2\}] \\ &= E[exp\{iu_1X_1 + iu_1\alpha Z + iu_2X_2 + iu_2\beta Z\}] \\ &= \psi_1(u_1)\psi_2(u_2)\psi_Z(\alpha u_1 + \beta u_2),\end{aligned} \tag{2.2}$$

where the last equality follows by independence and ψ_1, ψ_2 and ψ_Z are respectively the characteristic functions of X_1, X_2 and Z.
Obviously also the marginal distributions of W_1 and W_2 are infinitely divisible.

The bivariate distribution of $\boldsymbol{W}$ can be easily obtained conditioning with respect to Z.

$$\begin{aligned}F_{\boldsymbol{W}}(w_1,w_2) &= P(W_1 \le w_1, W_2 \le w_2) = P(X_1 + \alpha Z \le w_1, X_2 + \beta Z \le w_2) \\ &= \int_{\mathbb{R}} P(X_1 \le w_1 - \alpha z, X_2 \le w_2 - \beta z) dF_Z(z) \\ &= \int_{\mathbb{R}} F_{X_1}(w_1 - \alpha z) F_{X_2}(w_2 - \beta z) dF_Z(z),\end{aligned}$$

where $F_{X_i}, i = 1,2$ and F_Z are the distribution function of $X_i, i = 1,2$ and Z respectively. Furthermore, if we assume that all the random variables have a density - respectively f_1, f_2 and f_z -, the joint density of $\boldsymbol{W}$ exists and it is equal to

$$f_{\boldsymbol{W}}(w_1,w_2) = \int_{\mathbb{R}} f_{X_1}(w_1 - \alpha z) f_{X_2}(w_2 - \beta z) f_Z(z) dz.$$

The distribution of $\boldsymbol{W}$ is the starting point to define a Lévy process with correlated Lévy margins. The correlation between the marginal distributions and its expression as a function of the parameters of the model will be relevant for the applications. For this reason we will discuss it in details in the next section.

Define $\boldsymbol{G} = \{\boldsymbol{G}(t), t \ge 0\}$ the Lévy process such that $\mathcal{L}(\boldsymbol{G}(1)) = \mathcal{L}(\boldsymbol{W})$. The characteristic function of the process in each time t becomes:

$$\psi_{\boldsymbol{G}(t)}(u_1,u_2) = [\psi_{\boldsymbol{W}}(u_1,u_2)]^t = [\psi_1(u_1)\psi_2(u_2)\psi_Z(\alpha u_1 + \beta u_2)]^t. \tag{2.3}$$

The following is a technical remark to show a different way to construct a subordinator $\tilde{\boldsymbol{G}}$, so that $\mathcal{L}(\tilde{\boldsymbol{G}}(t)) = \mathcal{L}(\boldsymbol{G}(t))$ for each $t \in \mathbb{R}_+$. It will be useful in more than one proof in the sequel of the paper.

Remark 1. *Consider the independent Lévy processes* $\tilde{Z} = \{Z(t), t \geq 0\}$, $\tilde{X_1} = \{X_1(t), t \geq 0\}$ *and* $\tilde{X_2} = \{X_2(t), t \geq 0\}$, *defined by:*

$$\mathcal{L}(Z(1)) = \mathcal{L}(Z), \quad \mathcal{L}(X_1(1)) = \mathcal{L}(X_1), \quad \mathcal{L}(X_2(1)) = \mathcal{L}(X_2).$$

Define the bivariate process $\{\tilde{\boldsymbol{G}}(t), t \geq 0\}$

$$\tilde{\boldsymbol{G}}(t) = \begin{pmatrix} X_1(t) + \alpha Z(t) \\ X_2(t) + \beta Z(t) \end{pmatrix}. \tag{2.4}$$

For each t *fixed the characteristic function of* $\tilde{\boldsymbol{G}}(t)$ *follows from* (2.2) *and it is the following*

$$\begin{aligned} \psi_{\tilde{\boldsymbol{G}}(t)}(u_1, u_2) &= \psi_{X_1(t)}(u_1)\psi_{X_2(t)}(u_2)\psi_{Z(t)}(u_1\alpha + u_2\beta) \\ &= \psi_1(u_1)^t \psi_2(u_2)^t \psi_Z(u_1\alpha + u_2\beta)^t, \end{aligned}$$

which is the same as equation (2.3). *Therefore since the characteristic function uniquely determines the distribution we have that* $\mathcal{L}(\tilde{\boldsymbol{G}}(t)) = \mathcal{L}(\boldsymbol{G}(t))$ *for each* $t \geq 0$, *it means that for each fixed* t, $\boldsymbol{G}(t)$ *has the distribution of the sum of an independent random vector and a co-monotone one.*

The structure of the process $\boldsymbol{G}$ underlined in Remark 1 is crucial also to derive its Lévy measure.

We need to introduce some notation. Consider a set $B \in \mathcal{B}(\mathbb{R}^2)$ and $\Delta_{\alpha\beta} = \{(\alpha s, \beta s) : s \in \mathbb{R}_+\}$. Define $B_1 = \pi_1(B \cap \Delta_{\alpha\beta})$ and $B_2 = \pi_2(B \cap \Delta_{\alpha\beta})$, where $\pi_i, i = 1, 2$ are the projection of B on the coordinate axes. Obviously $\frac{B_1}{\alpha} = \{s \in \mathbb{R} : \alpha s \in B_1\}$ and analogously $\frac{B_2}{\beta}$, observe that by construction $\frac{B_1}{\alpha} = \frac{B_2}{\beta}$. Finally $B^* = B_1 \times B_2$.

Proposition 2.1. *Let* ν_1, ν_2, ν_Z *be respectively the Lévy measures of the processes* $\tilde{X_1}$, $\tilde{X_2}$, $\tilde{Z}$, *then the Lévy measure* $\nu_{\boldsymbol{G}}$ *of* $\boldsymbol{G}$ *satisfies*

$$\begin{aligned} \nu_{\boldsymbol{G}}(B) &= \nu_1(B_{10}) + \nu_2(B_{20}) + \nu_Z(\frac{B_1}{\alpha} \cap \frac{B_2}{\beta}) \\ &= \nu_1(B_{10}) + \nu_2(B_{20}) + \nu_Z(\frac{B_1}{\alpha}), \end{aligned} \tag{2.5}$$

where $B \in \mathcal{B}(\mathbb{R}^2 \setminus \{0\})$, $B_{10} = \{x \in \mathbb{R} : (x, 0) \in B\}$ *and* $B_{02} = \{y \in \mathbb{R} : (0, y) \in B\}$.

Proof. This proof is divided in different steps.

1. Let $\boldsymbol{X}$ and $\boldsymbol{Z}$ be respectively the bivariate processes $\boldsymbol{X} = \{(X_1(t), X_2(t))^T, t \geq 0\}$ and $\boldsymbol{Z} = \{(Z_1(t), Z_2(t))^T = (\alpha Z(t), \beta Z(t))^T, t \geq 0\}$.

   Let $\nu_{\boldsymbol{X}}$ be the Lévy measure associated to $\boldsymbol{X}$. Since the components $\tilde{X}_1$ and $\tilde{X}_2$ are independent, by Proposition 5.3 in [5] and Lemma 4.2 in [18] they never jump together, thus the support of $\nu_{\boldsymbol{X}}$ is $S_{\nu_{\boldsymbol{X}}} \subseteq \{(x, y) \in \mathbb{R}^2_+ : xy = 0\}$; moreover it holds:

$$\nu_{\boldsymbol{X}}(B) = \nu_1(B_{10}) + \nu_2(B_{20}). \tag{2.6}$$

2. Since the $\boldsymbol{G} = \boldsymbol{X} + \boldsymbol{Z}$, where $\boldsymbol{X}$ and $\boldsymbol{Z}$ are independent (see [5]) we obtain

$$\nu_{\boldsymbol{G}}(B) = \nu_{\boldsymbol{X}}(B) + \nu_{\boldsymbol{Z}}(B). \tag{2.7}$$

   It only remains to be determined the Lévy measure of $\boldsymbol{Z}$.

3. $\nu_{\boldsymbol{Z}}(B^*) = \nu_Z(\frac{B_1}{\alpha} \cap \frac{B_2}{\beta})$.

   In fact by definition

$$\nu_{\boldsymbol{Z}}(B^*) = E[\sharp\{t \in [0, 1] : \triangle \boldsymbol{Z}(t) \neq 0, \triangle \boldsymbol{Z}(t) \in B^*\}],$$

   where $\triangle \boldsymbol{Z}(t) = \boldsymbol{Z}(t) - \boldsymbol{Z}(t^-)$ is the jump at t. $\triangle \boldsymbol{Z}(t) \in B^*$ iff $\triangle Z_1(t) \in B_1$ and $\triangle Z_2(t) \in B_2$ iff $\triangle Z(t) \in \frac{B_1}{\alpha}$ and $\triangle Z(t) \in \frac{B_2}{\beta}$. Thus

$$\nu_{\boldsymbol{Z}}(B^*) = E[\sharp\{t \in [0, 1] : \triangle Z(t) \neq 0, \triangle Z(t) \in \frac{B_1}{\alpha} \cap \frac{B_2}{\beta}\}] = \nu_Z(\frac{B_1}{\alpha} \cap \frac{B_2}{\beta}).$$

4. For each $B \in \mathcal{B}(\mathbb{R}^2)$, $\nu_{\boldsymbol{Z}}(B) = \nu_{\boldsymbol{Z}}(B \cap \Delta_{\alpha\beta})$.

$$\begin{aligned}
\nu_{\boldsymbol{Z}}(B) &= E[\sharp\{t \in [0, 1] : \triangle \boldsymbol{Z}(t) \neq 0, \triangle \boldsymbol{Z}(t) \in B\}] \\
&= E[\sharp\{t \in [0, 1] : \triangle \boldsymbol{Z}(t) \neq 0, (\triangle Z_1(t), \triangle Z_2(t)) \in B\}] \\
&= E[\sharp\{t \in [0, 1] : \triangle Z(t) \neq 0, (\triangle \alpha Z(t), \triangle \beta Z(t)) \in B\}].
\end{aligned}$$

   But for each realization $(\triangle \alpha Z(t), \triangle \beta Z(t)) \in \Delta_{\alpha\beta}$, thus

$$\begin{aligned}
\nu_{\boldsymbol{Z}}(B) &= E[\sharp\{t \in [0, 1] : \triangle Z(t) \neq 0, (\triangle \alpha Z(t), \triangle \beta Z(t)) \in B\}] \\
&= E[\sharp\{t \in [0, 1] : \triangle Z(t) \neq 0, (\triangle \alpha Z(t), \triangle \beta Z(t)) \in B \cap \Delta_{\alpha\beta}\}] \\
&= E[\sharp\{t \in [0, 1] : \triangle \boldsymbol{Z}(t) \neq 0, \triangle \boldsymbol{Z}(t) \in B \cap \Delta_{\alpha\beta}\}] \\
&= \nu_{\boldsymbol{Z}}(B \cap \Delta_{\alpha\beta}).
\end{aligned}$$

5. $\nu_{\boldsymbol{Z}}(B) = \nu_Z(\frac{B_1}{\alpha} \cap \frac{B_2}{\beta})$. It suffices to observe that, by steps 3 and 4,

$$\nu_{\boldsymbol{Z}}(B) = \nu_{\boldsymbol{Z}}(B \cap \Delta_{\alpha\beta}) = \nu_{\boldsymbol{Z}}(B^* \cap \Delta_{\alpha\beta}) = \nu_{\boldsymbol{Z}}(B^*) = \nu_Z(\frac{B_1}{\alpha} \cap \frac{B_2}{\beta}). \tag{2.8}$$

By replacing (2.6) and (2.8) in equation (2.7) we get the assert.

□

The next proposition gives sufficient condition for $\boldsymbol{G}$ to be a subordinator. Under this condition we also find the characteristic exponent of $\boldsymbol{G}$ as a function of the processes $\boldsymbol{X}$ and $\boldsymbol{Z}$.

Proposition 2.2. *Let the processes $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{Z}$ be real subordinators, then the process $\boldsymbol{G}$ is a bivariate subordinator. Moreover the characteristic exponent $\Psi_{\boldsymbol{G}}$ of $\boldsymbol{G}$ satisfies:*

$$\begin{aligned}\Psi_{\boldsymbol{G}}(\boldsymbol{w}) &= i\langle \boldsymbol{m}, \boldsymbol{w}\rangle + \int_{\mathbb{R}^2_+} (e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle} - 1)\nu_{\boldsymbol{G}}(d\boldsymbol{z}) \\ &= i\langle \boldsymbol{m}, \boldsymbol{w}\rangle + \int_{\mathbb{R}^+} (e^{iw_1 z_1} - 1)\nu_1(dz_1) + \int_{\mathbb{R}^+} (e^{iw_2 z_2} - 1)\nu_2(dz_2) \\ &+ \int_{\mathbb{R}^+} (e^{i(\alpha w_1 + \beta w_2)s} - 1)\nu_Z(ds),\end{aligned} \tag{2.9}$$

where $\boldsymbol{m} \in \mathbb{R}^2$.

Proof. If $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{Z}$ are subordinators, then $\boldsymbol{G}$ takes values on $\mathbb{R}^2_+$, thus it is a bivariate subordinator. Therefore by (1.2):

$$\Psi_{\boldsymbol{G}}(\boldsymbol{w}) = i\langle \boldsymbol{m}, \boldsymbol{w}\rangle + \int_{\mathbb{R}^2} (e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle} - 1)\nu_{\boldsymbol{G}}(d\boldsymbol{z}),$$

where $\boldsymbol{m} \in \mathbb{R}^2_+$ and $\nu_{\boldsymbol{G}}$ is its Lévy measure. Consider now

$$\begin{aligned}\int_{\mathbb{R}^2_+} (e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle} - 1)\nu_{\boldsymbol{Z}}(d\boldsymbol{z}) &= \int_{\mathbb{R}^2_+} (e^{iw_1 z_1 + iw_2 z_2} - 1)\nu_{\boldsymbol{Z}}(d\boldsymbol{z}) \\ &= \int_{\mathbb{R}^2_+ \cap \Delta_{\alpha\beta}} (e^{iw_1 z_1 + iw_2 z_2} - 1)\nu_{\boldsymbol{Z}}(d\boldsymbol{z}) \\ &= \int_{\mathbb{R}_+} (e^{i(\alpha w_1 + \beta w_2)s} - 1)\nu_Z(ds)\end{aligned} \tag{2.10}$$

Therefore by equations (2.5) and (2.10) we have

$$\begin{aligned}\int_{\mathbb{R}_+^2}(e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle}-1)\nu_{\boldsymbol{G}}(d\boldsymbol{z}) &= \int_{\mathbb{R}_+^2}(e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle}-1)\nu_{\boldsymbol{X}}(d\boldsymbol{z})+\int_{\mathbb{R}_+^2}(e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle}-1)\nu_{\boldsymbol{Z}}(d\boldsymbol{z})\\ &= \int_{\mathbb{R}_+}(e^{iw_1z_1}-1)\nu_1(dz_1)+\int_{\mathbb{R}_+}(e^{iw_2z_2}-1)\nu_2(dz_2) \qquad (2.11)\\ &+ \int_{\mathbb{R}_+}(e^{i(\alpha w_1+\beta w_2)s}-1)\nu_Z(ds).\end{aligned}$$

□

Observe that the previous proposition implies that $\boldsymbol{G}$ has no diffusion component. Moreover by (2.3), $\Psi_{\boldsymbol{G}}(\boldsymbol{w}) = \Psi_1(w_1)+\Psi_2(w_2)+\Psi_Z(\alpha w_1+\beta w_2)$, where

$$\begin{aligned}\Psi_i(w) &= \textstyle\int_{\mathbb{R}_+}(e^{iwz}-1)\nu_i(dz)+in_iw, \quad i=1,2\\ \Psi_Z(w) &= \textstyle\int_{\mathbb{R}_+}(e^{iwz}-1)\nu_Z(dz)+in_zw \qquad (2.12)\end{aligned}$$

.

are respectively the exponent relative to the processes $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{Z}$. Therefore by equations (2.12) also

$$\begin{aligned}\int_{\mathbb{R}_+^2}(e^{i\langle \boldsymbol{w},\boldsymbol{z}\rangle}-1)\nu_{\boldsymbol{G}}(d\boldsymbol{z})+i\langle \boldsymbol{m},\boldsymbol{z}\rangle &= \int_{\mathbb{R}^+}(e^{iw_1z_1}-1)\nu_1(dz_1)+i(n_1z_1)\\ &+\int_{\mathbb{R}^+}(e^{iw_2z_2}-1)\nu_2(dz_2)+i(n_2z_2)\\ &+\int_{\mathbb{R}^+}(e^{i(\alpha w_1+\beta w_2)s}-1)\nu_Z(ds)+i(n_z(\alpha z_1+\beta z_2)),\end{aligned} \qquad (2.13)$$

which, together with (2.11) implies that $\boldsymbol{m} = (n_1+\alpha n_z, n_2+\beta n_z)$, in particular, if $n_i = 0$, $i = 1,2$ and $n_z = 0$ then $\boldsymbol{m} = 0$.

2.2 Step 2: a bivariate subordinator with Gamma margins

We specify the model introduced so that the process $\boldsymbol{G}$ has Gamma margins.

We start from the law of $\boldsymbol{W}$ so that its marginal distributions are Gamma. We obtain a generalization of the bivariate McKay Gamma distribution studied in [1] and also of the one studied by Martinelli [12].

Define $\boldsymbol{W}$ as in (2.1), where Z is a fixed random variable such that:

$$\mathcal{L}(Z) = \Gamma(a,b),\ a,b>0 \qquad (2.14)$$

and

$$\mathcal{L}(X_1) = \Gamma(a_1, \tfrac{b}{\alpha}) \quad \mathcal{L}(X_2) = \Gamma(a_2, \tfrac{b}{\beta}), \quad a_1, a_2, \tfrac{b}{\alpha}, \tfrac{b}{\beta} > 0. \tag{2.15}$$

Observe that from the fact $b > 0$ it also follows that α and β have to be positive. By the Gamma distribution properties $\mathcal{L}(\alpha Z) = \Gamma(a, \frac{b}{\alpha})$, $\mathcal{L}(W_1) = \Gamma(\tilde{a}_1, \frac{b}{\alpha})$ and $\mathcal{L}(W_2) = \Gamma(\tilde{a}_2, \frac{b}{\beta})$, with $\tilde{a}_i = a_i + a$, $i = 1, 2$. Since $X_i, i = 1, 2$ and Z are infinitely divisible the assumptions made in the previous section hold and the random vector $\boldsymbol{W}$ is also jointly infinitely divisible.

We now impose some restrictions on the parameters of X_1 and X_2 so that W_1 and W_2 satisfy $E[W_i] = 1, i = 1, 2$. Actually this last requirement is due to the final model, and therefore will be discussed in Section 2.3. Please notice that (2.14) and (2.15) imply $E[Z] = \frac{a}{b}$, $E[X_1] = \frac{a_1\alpha}{b}$ and $E[X_2] = \frac{a_2\beta}{b}$. From (2.1)

$$X_1 = W_1 - \alpha Z, \quad X_2 = W_2 - \beta Z.$$

The condition $E[W_1] = 1$ implies $\frac{a_1}{\frac{b}{\alpha}} = 1 - \alpha\frac{a}{b}$ and $a_1 = \frac{b}{\alpha} - a$, in the same way we obtain $a_2 = \frac{b}{\beta} - a$. Therefore the parameters of X_1 and X_2 are positive iff we impose the following restriction for α and β:

$$0 < \alpha < \tfrac{b}{a} \quad 0 < \beta < \tfrac{b}{a}. \tag{2.16}$$

Furthermore we have $\tilde{a}_1 = a_1 + a = \frac{b}{\alpha}$; an analogous argument applies, by replacing $\tilde{a}_1$ by $\tilde{a}_2$. We get $\mathcal{L}(W_1) = \Gamma(\frac{b}{\alpha}, \frac{b}{\alpha})$ and $\mathcal{L}(W_2) = \Gamma(\frac{b}{\beta}, \frac{b}{\beta})$.

We have proved the following

Proposition 2.3. *Let a, b, α, β be positive real parameters which satisfy conditions* (2.16)*. Let $\mathcal{L}(X_1) = \Gamma(\frac{b}{\alpha} - a, \frac{b}{\alpha})$ and $\mathcal{L}(X_2) = \Gamma(\frac{b}{\beta} - a, \frac{b}{\beta})$ be two independent random variables and let them be also independent from a random variable such that $\mathcal{L}(Z) = \Gamma(a, b)$. Define $\boldsymbol{W}$ as in* (2.1)*, then $\mathcal{L}(W_1) = \Gamma(\frac{b}{\alpha}, \frac{b}{\alpha})$ and $\mathcal{L}(W_2) = \Gamma(\frac{b}{\beta}, \frac{b}{\beta})$. The random vector $\boldsymbol{W}$ is jointly infinitely divisible.*

We name $\alpha\beta$-Gamma the distribution of $\boldsymbol{W}$ as defined in Proposition 2.3.

Since the characteristic function is fundamental in studying Lévy processes, we recall the characteristic functions of the margins, that are Gamma random variables:

$$\psi_{W_1}(u) = (1 - \tfrac{iu\alpha}{b})^{-\frac{b}{\alpha}} \quad \psi_{W_2}(u) = (1 - \tfrac{iu\beta}{b})^{-\frac{b}{\beta}},$$

the characteristic function of $\boldsymbol{W}$, according to (2.17), is

$$\begin{aligned}\psi_{\boldsymbol{W}}(u_1,u_2) &= \psi_1(u_1)\psi_2(u_2)\psi_Z(u_1\alpha+u_2\beta)\\ &= (1-\frac{iu_1\alpha}{b})^{-(\frac{b}{\alpha}-a)}(1-\frac{iu_2\beta}{b})^{-(\frac{b}{\beta}-a)}(1-\frac{iu_1\alpha+iu_2\beta}{b})^{-a}.\end{aligned} \tag{2.17}$$

Observe that both the joint and the marginal characteristic functions do not depend on the parameters α, β, b separately but only through $\frac{\alpha}{b}$ and $\frac{\beta}{b}$.

Before going on with the construction we make some considerations about the distribution of $\boldsymbol{W}$ when the parameters go to the limit of their admissible values. The cases between brackets are similar to the others and we omit them. The symbol δ_a stands for the distribution concentrated at $a \in \mathbb{R}$.

1. $\alpha, \beta \to 0$.
   By equation (2.1) $W_i = X_i$ for $i = 1, 2$. Notice that

   $$V(X_1) = \tfrac{\alpha}{b} - a(\tfrac{\alpha}{b})^2 \quad V(X_2) = \tfrac{\beta}{b} - a(\tfrac{\beta}{b})^2. \tag{2.18}$$

   Easily $V(X_1) \to 0$, that together with $E[X_1] = 1$ implies $\mathcal{L}(X_i) = \delta_1$ for $i = 1, 2$. Furthermore $\boldsymbol{W}$ has independent components.

2. $\alpha \to 0, \quad \beta \neq 0, \frac{b}{a}$ $[\beta \to 0, \alpha \neq 0, \frac{b}{a}]$.
   As in previous case $W_1 = X_1$ and $\mathcal{L}(W_1) = \mathcal{L}(X_1) = \delta_1$, while $\mathcal{L}(W_2) = \mathcal{L}(X_2 + \beta Z)$. The components are independent.

3. $\alpha, \beta \to \frac{b}{a}$.
   Easily $E[X_i] = 0$ for $i = 1, 2$ and by (2.18) $V(X_i) \to 0$ for $i = 1, 2$ thus $\mathcal{L}(X_i) = \delta_0$. Therefore $\mathcal{L}(W_1) = \mathcal{L}(\alpha Z) = \Gamma(a, a)$ and $\mathcal{L}(W_2) = \mathcal{L}(\beta Z) = \Gamma(a, a)$. Obviously the linear correlation coefficient is 1. We will see at the end of our construction that under this conditions our model becomes the one considered in Luciano and Schoutens [8].

4. $\alpha \to \frac{b}{a}, \quad \beta \neq \frac{b}{a}, 0$ $[\beta \to \frac{b}{a}, \quad \alpha \neq \frac{b}{a}, 0]$.
   We have $\mathcal{L}(X_1) = \delta_0$, and $\mathcal{L}(W_1) = \mathcal{L}(\alpha Z)$ and $\mathcal{L}(W_2) = \mathcal{L}(X_2 + \beta Z)$, we find the the model studied by Martinelli [12].

5. $\alpha \to 0, \quad \beta \to \frac{b}{a}$ $[\beta \to 0, \quad \alpha \to \frac{b}{a}]$.
   As in the previous case $\mathcal{L}(W_1) = \mathcal{L}(X_1) = \delta_1$ and $\mathcal{L}(X_2) = \delta_0$, while $\mathcal{L}(W_2) = \mathcal{L}(\beta Z)$. The components are independent.

Observe that $\boldsymbol{W}$ has independent components iff $\alpha = 0$ or $\beta = 0$ and in both cases we have one degenerate random variable at least. The reason is that the variances of X_1 and X_2 are proportional to α and β respectively.

Define $\boldsymbol{G} = \{\boldsymbol{G}(t), t > 0\}$ to be the Lévy process so that $\mathcal{L}(\boldsymbol{G}(1)) = \mathcal{L}(\boldsymbol{W})$, where $\boldsymbol{W}$ be the infinitely divisible random vector introduced in Proposition 2.3.

The following corollary is a direct consequence of Remark 1 and of Proposition 2.3:

Corollary 2.1. *Let $\boldsymbol{G} = \{\boldsymbol{G}(t), t \geq 0\}$ be the Lévy process associated to the distribution of $\boldsymbol{W}$ defined in Proposition 2.3, then*

$$\mathcal{L}(G_1(t)) = \Gamma(\tfrac{tb}{\alpha}, \tfrac{b}{\alpha}) \quad \mathcal{L}(G_2(t)) = \Gamma(\tfrac{tb}{\beta}, \tfrac{b}{\beta}),$$

and they have the following means and variances:

$$\begin{array}{cc} E[G_1(t)] = t & E[G_2(t)] = t \\ V(G_1(t)) = \frac{t\alpha}{b} & V(G_2(t)) = \frac{t\beta}{b}. \end{array}$$

We name the process $\boldsymbol{G}$ $\alpha\beta$-Gamma process.

Gamma processes are real subordinators; therefore, as observed in the more general setting of the previous step, $\boldsymbol{G}$ results to be a bivariate subordinator. The Lévy measures of $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{Z}$ are known, as they are Gamma processes (see the preliminary section). The Lévy measure of $\boldsymbol{G}$ is completely determined by equation (2.5) as a function of the Lévy measures of $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{Z}$. Let $B \in \mathbf{B}(\mathbb{R}^2)$ and B_{10}, B_{02}, B_1, B_2 defined as in Proposition 2.1, we have

$$\begin{aligned} \nu_{\boldsymbol{G}}(B) &= \nu_1(B_{10}) + \nu_2(B_{02}) + \nu_Z(\frac{B_1}{\alpha} \cap \frac{B_2}{\beta}) \\ &= \int_{B_{10}} (\frac{b}{\alpha} - a) exp(-\frac{b}{\alpha}x) x^{-1} 1_{(0,+\infty)}(x) dx + \int_{B_{02}} (\frac{b}{\beta} - a) exp(-\frac{b}{\beta}x) x^{-1} 1_{(0,+\infty)}(x) dx \\ &+ \int_{\frac{B_1}{\alpha} \cap \frac{B_2}{\beta}} a exp(-bx) x^{-1} 1_{(0,+\infty)}(x) dx. \end{aligned}$$

The characteristic exponent $\boldsymbol{G}$ follows from (2.9) with $\boldsymbol{m} = 0$: as observed in the preliminary section the parameters n_1, n_2, n_z in equation (2.12) are all zero because X_1, X_2 and Z are Gamma processes. Moreover the expression of $\Psi_{\boldsymbol{G}}$ can also be easily obtained by the following:

$$\begin{aligned}\Psi_{\boldsymbol{G}}(\boldsymbol{z}) &= \Psi_1(z_1) + \Psi_2(z_2) + \Psi_z(\alpha z_1 + \beta z_2) \\ &= \log[\psi_1(z_1)] + \log[\psi_2(z_2)] + \log[\psi_Z(\alpha z_1 + \beta z_2)] \\ &= \log[(1 - \frac{iz_1\alpha}{b})^{-(\frac{b}{\alpha}-a)}] + \log[(1 - \frac{iz_2\beta}{b})^{-(\frac{b}{\beta}-a)}] + \log[(1 - \frac{iz_1\alpha + iz_2\beta}{b})^{-a}],\end{aligned} \tag{2.19}$$

$z \in \mathbb{R}^2$.

2.3 Step 3: The model

This last step consists in the definition of our bivariate extension of the models in [10] and in [8] by means of a time change in a Brownian motion through the subordinator $\boldsymbol{G}$.

Let $B_1 = \{B_1(t), t \geq 0\}$ and $B_2 = \{B_2(t), t \geq 0\}$ be two independent standard Brownian motions. Consider the process $\tilde{\boldsymbol{B}} = \{\tilde{\boldsymbol{B}}(t), t > 0\}$

$$\tilde{\boldsymbol{B}}(t) = \begin{pmatrix} \tilde{B}_1(t) \\ \tilde{B}_2(t) \end{pmatrix} = \begin{pmatrix} \mu_1 t + \sigma_1 B_1(t) \\ \mu_2 t + \sigma_2 B_2(t) \end{pmatrix}, \tag{2.20}$$

the Lévy triplet of $\tilde{\boldsymbol{B}}$ is obviously $(\mu, \Sigma, 0)$, where $\mu = (\mu_1, \mu_2)$ and $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$.

Definition 2.1. *The bivariate process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ is defined by the equations:*

$$\boldsymbol{Y}(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = \begin{pmatrix} \mu_1 G_1(t) + \sigma_1 B_1(G_1(t)) \\ \mu_2 G_2(t) + \sigma_2 B_2(G_2(t)) \end{pmatrix}, \tag{2.21}$$

where $\tilde{\boldsymbol{B}}$ is given by (2.20) *and $\boldsymbol{G}$ is an $\alpha\beta-$Gamma process independent from $\tilde{\boldsymbol{B}}$. The process $\boldsymbol{Y}$ is named $\alpha\beta$-Variance Gamma process.*

This definition clarifies the assumption on the mean values of the marginal distribution of $\boldsymbol{W}$ made in the second step. We required that $E[W_1] = E[W_2] = 1$, that obviously imply $E[G_1(t)] = t$ and $E[G_2(t)] = t$. In this environment the interpretation is that the changed time $\boldsymbol{G}(t)$ goes in mean like the real one, t.

We prove that the $\alpha\beta$-Variance Gamma is a pure jump Lévy process. We find a closed form for its characteristic function, and we investigate its Lévy triplet.

The following result is a direct consequence of Theorem 1.1 (for the general statement and its proof see [2]).

We have to consider here the multi-parameter process $\{\tilde{\boldsymbol{B}}(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^2_+\}$ defined by $\tilde{\boldsymbol{B}}(\boldsymbol{s}) = (\tilde{B}_1(s_1), \tilde{B}_2(s_2)),\ \boldsymbol{s} \in \mathbb{R}^2_+\}$.

Theorem 2.1. *The process $\boldsymbol{Y}$ defined in (2.21) is a Lévy process. Its Lévy triplet $(\gamma_{\boldsymbol{Y}}, \Sigma_{\boldsymbol{Y}}, \nu_{\boldsymbol{Y}})$ is as follows:*

$$\begin{gathered}\gamma_{\boldsymbol{Y}} = \int_{\mathbb{R}^2_+} \nu_{\boldsymbol{G}}(d\boldsymbol{s}) \int_{|\boldsymbol{x}|\leq 1} \boldsymbol{x}\rho_{\boldsymbol{s}}(d\boldsymbol{x}),\\ \Sigma_{\boldsymbol{Y}} = \boldsymbol{0},\\ \nu_{\boldsymbol{Y}}(B) = \int_{\mathbb{R}^2_+} \rho_{\boldsymbol{s}}(B)\nu_{\boldsymbol{G}}(d\boldsymbol{s}),\end{gathered}$$

where $\rho_s = \mathcal{L}(\tilde{\boldsymbol{B}}(\boldsymbol{s}))$, $s \in \mathbb{R}^2_+$, $\boldsymbol{x} = (x_1, x_2)^T$ and $B \in \mathbb{R}^2 \setminus \{0\}$. Its characteristic function is the following [1]

$$\begin{aligned}\psi_{\boldsymbol{Y}(t)}(u_1, u_2) = (1 - \frac{\alpha(i\mu_1 u_1 - \frac{1}{2}\sigma_1^2 u_1^2)}{b})^{-t(\frac{b}{\alpha}-a)}(1 - \frac{\beta(i\mu_2 u_2 - \frac{1}{2}\sigma_2^2 u_2^2)}{b})^{-t(\frac{b}{\beta}-a)}\\ \cdot (1 - \frac{\alpha(i\mu_1 u_1 - \frac{1}{2}\sigma_1^2 u_1^2) + \beta(i\mu_2 u_2 - \frac{1}{2}\sigma_2^2 u_2^2)}{b})^{-ta}.\end{aligned} \tag{2.22}$$

Proof. By definition the process $\tilde{\boldsymbol{B}}$ is a Lévy process with independent components and $\boldsymbol{G}$ is a bivariate subordinator, therefore Theorem 1.1 guarantees that $\boldsymbol{Y}$ is also a Lévy process. Let now $\Sigma_{\boldsymbol{Y}}$ be the diffusion matrix of $\boldsymbol{Y}$. By Theorem 1.1, it holds

$$\Sigma_{\boldsymbol{Y}} = diag(m_1\sigma_1, m_2\sigma_2),$$

where $\boldsymbol{m} = (m_1, m_2)$ is the parameter in (2.9). We observed that if $\boldsymbol{G}$ is an $\alpha\beta$- Gamma process then $\boldsymbol{m} = 0$. We have $\Sigma = \boldsymbol{0}$ and the assertion. The expression for the Lévy measure and the drift are immediate, applying Theorem 1.1. The characteristic function of $\boldsymbol{Y}$ is given by the following:

$$\psi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = E[e^{\langle \boldsymbol{u}, \boldsymbol{Y}(t)\rangle}] = exp(t\Psi_{\boldsymbol{G}}(\log \psi_{\tilde{\boldsymbol{B}}}(\boldsymbol{u}))),$$

where for any $\boldsymbol{w} = (w_1, w_2) \in \mathbb{C}^2$ with $Re(w_i) \leq 0$, $\Psi_{\boldsymbol{G}}(\boldsymbol{w})$ can be easily deduced by equation (2.19). Since $\psi_{\tilde{\boldsymbol{B}}}$ is the characteristic function of a Brownian motion with drift μ and covariance matrix Σ, (2.22) follows by an easy computation. □

[1] It is easy to find directly the Laplace transform of the process, that is

$$\begin{aligned}\varphi_{\boldsymbol{Y}(t)}(u_1, u_2) = E[exp\{-u_1 X(t) - u_2 X(t)\}] = (\varphi_{G(1)}(\mu_1 u_1 + \frac{1}{2}\sigma_1^2 u_1^2, m u_2 u_2 + \frac{1}{2}\sigma_2^2 u_2^2))^t\\ = (1 - \frac{\alpha(\mu_1 u_1 + \frac{1}{2}\sigma_1^2 u_1^2)}{b})^{-t(\frac{b}{\alpha}-a)}(1 - \frac{\beta(\mu_2 u_2 + \frac{1}{2}\sigma_2^2 u_2^2)}{b})^{-t(\frac{b}{\beta}-a)}\\ \cdot (1 - \frac{\alpha(\mu_1 u_1 + \frac{1}{2}\sigma_1^2 u_1^2) + \beta(\mu_2 u_2 + \frac{1}{2}\sigma_2^2 u_2^2)}{b})^{-ta}.\end{aligned}$$

We underline that in the previous proof the independence between the processes B_1 and B_2 is necessary in order to apply Theorem 1.1.

Observe that the process $\boldsymbol{Y}$ defined in (2.21) is a pure jump process with both positive and negative jumps, as we can infer from the Lévy measure $\nu_{\boldsymbol{Y}}$. Moreover the process $\boldsymbol{Y}$ has a jump in t iff the process $\boldsymbol{G}$ has. In this case $\boldsymbol{G}(t) \neq \boldsymbol{G}(t^-)$. Assume $\boldsymbol{G}(t) = \boldsymbol{s}$ and $\boldsymbol{G}(t^-) = \boldsymbol{s}' \preceq \boldsymbol{s}$, $\boldsymbol{s} \neq \boldsymbol{s}'$. Under this condition the process $\boldsymbol{Y}$ has a jump in t with probability 1, in fact $\mathcal{L}(\Delta \boldsymbol{Y}(t) | \boldsymbol{G}(t) = \boldsymbol{s}) = \mathcal{L}(\tilde{\boldsymbol{B}}(\boldsymbol{s}) - \tilde{\boldsymbol{B}}(\boldsymbol{s}'))$. The marginal distributions of the jumps become $\mathcal{L}(\tilde{B}_1(s_1) - \tilde{B}_1(s_1')) = N(\mu_1(s_1 - s_1'), \sigma_1^2(s_1 - s_1'))$ and $\mathcal{L}(\tilde{B}_2(s_2) - \tilde{B}_2(s_2')) = N(\mu_2(s_2 - s_2'), \sigma_2^2(s_2 - s_2'))$. The multi-parameter is thus involved in the jump distributions: specifically, if the components of the Browinan motion have the same distribution ($\mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$), the jumps of $Y_1(t)$ and $Y_2(t)$ may have different laws. They have the same distribution if and only if $\Delta \boldsymbol{G}_1(t) = \Delta \boldsymbol{G}_2(t)$. Furthermore the two components with positive probability don't jump together: it suffices that the components of $\boldsymbol{G}$ don't jump in the same t (for example in t the only jump is due to $X_1(t)$).

Consider now the limit case $\alpha, \beta \to \frac{b}{a}$. As observed in step 2, W_1 and W_2 are co-monotone, moreover $W_1 = \alpha Z = \beta Z = W_2$ and their distribution is $\Gamma(a, a)$. As a consequence the margins of the process $\boldsymbol{G}$ satisfy $G_1(t) = \alpha Z(t) = \beta Z(t) = G_2(t)$, for each t which is equivalent to have a univariate Gamma subordinator. We find the model studied in Luciano and Schoutens [8]. Obviously also in this limit case $\boldsymbol{Y}$ is a pure jump process. Suppose that the subordinator has a jump in t, it means $\Gamma(t^-) = s' < s = \Gamma(t)$. In this case if $\mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$ the jumps of the two margins always have the same law. Moreover the two components always jump together.

Theorem 2.1 can be proved in a more general setting. We can extend the $\alpha\beta-Gamma$ subordinator $\boldsymbol{G}$ on $\mathbb{R}^d$ defining for each $i = 1, ..., d$

$$G_i(t) = X_i(t) + \alpha_i Z(t), \ t \geq 0.$$

If $\{\tilde{\boldsymbol{B}}_i, i = 1, ..., d\}$ are d independent Brownian motion on $\mathbb{R}^{n_i}$, then by Theorem 3.3 in [2] the subordinated process $\boldsymbol{Y}(t) = (\tilde{\boldsymbol{B}}_1(G_1(t)), ..., \tilde{\boldsymbol{B}}_d(G_d(t)))$ is a Lévy process, where each marginal is a Variance Gamma process. For each $i = 1, ..., d$, $\tilde{\boldsymbol{B}}_i$ is a multivariate Variance Gamma process with correlated Brownian motions and a real subordinator.

3 The dependence structure

We have introduced a bivariate Lévy model with correlated margins. Here we analyze the correlation of the process as a function of the parameters α and β and we compare our model with the bivariate Variance Gamma process studied in [8]. We do it step by step starting from the correlation of $\boldsymbol{W}$ in the general case. Both the covariance and the correlation coefficient $\rho_{\boldsymbol{W}}$ depend on α and β.

$$Cov(W_1, W_2) = E[W_1 W_2] - E[W_1]E[W_2] = \alpha\beta V(Z).$$

Since $V(W_1) = V(X_1) + \alpha^2 V(Z)$ and $V(W_2) = V(X_2) + \beta^2 V(Z)$ we have

$$\begin{aligned}\rho_{\boldsymbol{W}} &= \frac{\alpha\beta V(Z)}{\sqrt{[V(X_1) + \alpha^2 V(Z)][V(X_2) + \beta^2 V(Z)]}} \\ &= \sqrt{\frac{1}{[\frac{V(X_1)}{\alpha^2 V(Z)} + 1][\frac{V(X_2)}{\beta^2 V(Z)} + 1]}}.\end{aligned}$$

We consider $\boldsymbol{W}$ as specified in Proposition 2.3. The main moments of $\boldsymbol{W}$ are:

$$\begin{array}{ll} E[W_1] = 1 & E[W_2] = 1 \\ V[W_1] = \frac{\alpha}{b} & V[W_2] = \frac{\beta}{b}. \end{array}$$

The covariance is

$$Cov(W_1, W_2) = \alpha\beta V(Z) = \alpha\beta\frac{a}{b^2},$$

and the linear correlation coefficient is:

$$\rho_{\boldsymbol{W}} = \frac{Cov(W_1, W_2)}{\sqrt{V(W_1)V(W_2)}} = \frac{\alpha\beta\frac{a}{b^2}}{\sqrt{\frac{\alpha}{b}\frac{\beta}{b}}} = \frac{a}{b}\sqrt{\alpha\beta}.$$

It is also easy to verify that the correlation coefficient of the process $\boldsymbol{G}$ doesn't depend on t is increasing in α and β and is the same as the correlation at time 1:

$$\rho_{\boldsymbol{G}(t)} = \frac{a}{b}\sqrt{\alpha\beta}.$$

The parameters of $\boldsymbol{G}(t)$ are α, β, b, a. They are all involved to determine the correlation coefficient $\rho_{\boldsymbol{G}}$. It is interesting to observe that $\rho_{\boldsymbol{G}}$ depends on α, β and b only through the ratios $\frac{\alpha}{b}$ and $\frac{\beta}{b}$. It implies that if we fix the margins $G_1(t)$ and $G_2(t)$ (which is equivalent to fixing the previous ratios), $\rho_{\boldsymbol{G}}$ is only a function of a. Anyway because of the constraint of the model the maximal correlation $\rho_{\boldsymbol{G}} = 1$ is reached iff $\frac{\alpha}{b} = \frac{\beta}{b} = \frac{1}{a}$.

We now calculate the main moments of the process $\boldsymbol{Y}$.

$$\begin{array}{cc} E[Y_1(t)] = \mu_1 t & E[Y_2(t)] = \mu_2 t \\ V(Y_1(t)) = t(\sigma_1^2 + \mu_1^2 \frac{\alpha}{b}) & V(Y_2(t)) = t(\sigma_2^2 + \mu_2^2 \frac{\beta}{b}). \end{array}$$

The covariance of the process at time t is:

$$cov[Y_1(t), Y_2(t)] = \mu_1 \mu_2 cov[G_1(t), G_2(t)] = t\mu_1\mu_2 \frac{a}{b^2}\alpha\beta.$$

Finally the linear correlation coefficient is

$$\rho_{\boldsymbol{Y}(t)} = \frac{cov[Y_1(t), Y_2(t)]}{\sqrt{V(Y_1(t))V(Y_2(t))}} = \frac{\mu_1\mu_2\alpha\beta\frac{a}{b^2}}{\sqrt{(\sigma_1^2 + \mu_1^2\frac{\alpha}{b})(\sigma_2^2 + \mu_2^2\frac{\beta}{b})}} = \frac{\mu_1\mu_2\alpha\beta a}{b\sqrt{(b\sigma_1^2 + \mu_1^2\alpha)(b\sigma_2^2 + \mu_2^2\beta)}},$$

which is independent of t and is increasing both in α and β. The correlation of the process involves all the parameters, and for any couple of fixed marginal distributions it is a function only of a. If $\mu_1, \mu_2 = 0$ the correlation is zero, even if the process is clearly dependent. Finally $\rho_{\boldsymbol{Y}(t)} > 0$ if and only if $\mu_1\mu_2 > 0$. We find the case of Luciano and Schoutens iff $\frac{\alpha}{b} = \frac{\beta}{b} = \frac{1}{a}$, in fact in their model once fixed the marginal distribution the correlation coefficient is uniquely determinate. Also in this particular case $\mu_1, \mu_2 = 0$ imply $\rho_{\boldsymbol{Y}(t)} = 0$ and $\rho_{\boldsymbol{Y}(t)} > 0$ if and only if $\mu_1\mu_2 > 0$.

The following proposition concerns the dependence structure of the subordinator $\boldsymbol{G}$ for a fixed time, say t. Its interest is mainly due to the fact that in this model the subordinator has always positively dependent components.

We first need the following (for a complete overview see [15] and [14]):

Definition 3.1. *1. Given two random variables X and Y, with distribution functions F_X and F_Y respectively, X is said to be smaller than Y in the usual stochastic order (denoted $X \leq_{st} Y$) if*

$$\bar{F}_X(x) \leq \bar{F}_Y(x), \forall x \in \mathbb{R},$$

where $\bar{F}_X(x) = P(X > x)$, $\forall t \in \mathbb{R}$.

2. A random vector $\boldsymbol{A}$ is said to be stochastically increasing in the random vector $\boldsymbol{B}$ if the conditional distribution of $\boldsymbol{A}$ given $\boldsymbol{B} = \boldsymbol{b}$ is increasing in $\boldsymbol{b}$ in the usual stochastic order.

3. A random vector $\boldsymbol{X} = (X_1, X_2)$ is said to be positively associated (shortly PA*) if*

$$cov(\Phi_1(X_1, X_2), \Phi_2(X_1, X_2)) \geq 0 \tag{3.1}$$

for all increasing functions Φ_1 and Φ_2 for which the covariance is well-defined.

Proposition 3.1. *For each value of the parameters $\boldsymbol{G}(t)$ is PA.*

Proof. Let $\tilde{\boldsymbol{G}}(t) = \boldsymbol{X}(t) + \boldsymbol{Z}(t)$, where $\boldsymbol{Z}(t) = (Z_1(t), Z_2(t))^T = (\alpha Z(t), \beta Z(t))^T$ and $\boldsymbol{X}(t) = (X_1(t), X_2(t))^T$, be defined as in Remark 1. The random vector $\boldsymbol{Z}(t)$ is co-monotone, then obviously PA.
$\tilde{\boldsymbol{G}}(t)$ is stochastically increasing in $\boldsymbol{Z}(t)$: since $\boldsymbol{X}(t)$ and $\boldsymbol{Z}(t)$ are independent then $\mathcal{L}(\tilde{\boldsymbol{G}}(t)|\boldsymbol{Z}(t) = \boldsymbol{z}) = \mathcal{L}(\boldsymbol{X}(t) + \boldsymbol{z})$ and it is easy to verify that $\boldsymbol{z} \leq \boldsymbol{z}' \Rightarrow \boldsymbol{X}(t) + \boldsymbol{z} \leq_{st} \boldsymbol{X}(t) + \boldsymbol{z}'$. Moreover for each $\boldsymbol{z} \in \mathbb{R}^2$, $\boldsymbol{X}(t) + \boldsymbol{z}$ is PA because it has independent components. By Proposition 2.1 in [3] we conclude $\tilde{\boldsymbol{G}}(t) = \boldsymbol{X}(t) + \boldsymbol{Z}(t)$ is PA. The assertion follows because $\mathcal{L}(\boldsymbol{G}(t)) = \mathcal{L}(\tilde{\boldsymbol{G}}(t))$. □

4 A financial model

As mentioned in the introduction Variance Gamma processes are widely used in finance. We have proved that the $\alpha\beta$-Variance Gamma process is a generalization of the process that Luciano and Schoutens proposed as a price model. For this reason we propose a price model based on our Lévy process. As in [8], we define the price process to be the exponential of the process $\boldsymbol{Y}$:

$$\tilde{\boldsymbol{S}}(t) = \tilde{\boldsymbol{S}}(0) exp(\boldsymbol{Y}(t)), \quad t \geq 0,$$

the i-th component of S is

$$\tilde{S}_i(t) = \tilde{S}_i(0) exp(Y_i(t)), \quad t \geq 0.$$

Cont and Tankov (2004)[5] (Section 9.5) list the exponential-Lévy model that are arbitrage free. Our model belongs to their list, since it has both positive and negative jumps. Since this is an exponential-Lévy model which is arbitrage-free, there exists an equivalent martingale measure Q, i.e. a probability measure equivalent to the real world one, such that the discounted marginal processes are Q-martingales. The model however belongs to the class of incomplete models (see Schoutens [16] (6.2.1)): the equivalent martingale measure is not unique. The stock price model $\tilde{\boldsymbol{S}}$ is a generalization of the one studied by Luciano and Schoutens, which belongs to the same class of exponential Lévy models. This motives our interest here in comparing them on the light of their financial application.

A relevant good property of our model that can be deduced from the characteristic function of $\boldsymbol{Y}$ is that the in the asymmetric case ($\mu_1, \mu_2 \neq 0$) the process is not elliptically distributed. The same holds for the Luciano and Schoutens' one, while the first multivariate one proposed by Madan and Seneta is elliptically distributed, as one can infer from its characteristic function (see [10]) [2].

The dependence structure of the process has also its interest in financial application. The characteristic of the dependence of the Variance Gamma process with a common gamma subordiator is that once found the parameters of the marginal processes the correlation coefficient is uniquely determinate. The main contribution of our generalization with respect to correlation is that even if the parameters of the marginal processes are given, the coefficient depends on one more parameter: a. Therefore changing a we can modify the correlation of the process, without modifying the marginal distributions of the process. However in this way we cannot find the maximal correlation for the subordinator. Indeed, as observed in the previous section, the constraints of the model impose that the limit case of $\rho_{\boldsymbol{G}} = 1$ is obtained iff $\alpha = \beta$ and this is nothing but the model studied by Luciano and Schoutens [3].

5 Simulations

In this section we simulate both the subordinator and the process for different values of the parameter. We are interested in comparing processes with different correlation coefficient. Since the correlation depend on α, β and b only trough the ratios $\frac{\alpha}{b}$ and $\frac{\beta}{b}$ we fix $b = 1$. We compare two cases in which $\boldsymbol{G}$ has correlated margins with $\rho_{\boldsymbol{G}} < 1$, to see how correlation change as a function of the parameters, in particular we fix the marginal distributions and vary the parameter a. Then we simulate the limit case with $\rho_{\boldsymbol{G}} = 1$ and $\alpha = \beta$, the one investigated in [8]. The case in which $\boldsymbol{G}$ has independent

[2] A n-dimensional random vector $\boldsymbol{X}$ has an elliptical distribution if there is $\mu \in \mathbb{R}^n$ and a non negative definite, symmetric $n \times n$ matrix Σ such that the characteristic function of $\boldsymbol{X} - \mu$ is a function of the quadratic form $t^T \Sigma t$. A typical example is the Normal distribution. The margins of an elliptical distributions are elliptical and of the same type, and this is a limit for financial applications. Moreover since they are radially symmetric the coefficient of upper and lower tail dependence are equal. For a discussion on this matter see [6].

[3] We recall that a bivariate model for pricing based on correlated Lévy processes with Variance Gamma margins has also been proposed by Cont and Tankov [5]. It is constructed by means of Lévy copulas. They fix two univariate Variance Gamma processes and specify their dependence using a Lévy copula F. They discuss this technique with an example.

component is meaningless, in fact as observed in Section 3 at least one of the components degenerates.

We start with the scatter plots of the processes $\boldsymbol{Y}$ at time $t = 1$:

- We simulate N realizations from the independent $\mathcal{L}(X_1) = \Gamma((\frac{b}{\alpha} - a), \frac{b}{\alpha})$, $\mathcal{L}(X_2) = \Gamma((\frac{b}{\beta} - a), \frac{b}{\beta})$, $\mathcal{L}(Z) = \Gamma(a, b)$, let them be respectively x_1^n, x_2^n, z^n for $n = 1, ..., N$;
- We obtain N realizations (w_1^n, w_2^n) of $\boldsymbol{W}$ through the relations $W_1 = X_1 + Z$ and $W_2 = X_2 + Z$;
- We generate N independent random numbers $\{M_1(n) : n = 1, ...N\}$ extracted from a variable M_1 so that $\mathcal{L}(M_1) = N(0, w_1^n)$ and N independent random numbers $\{M_2(n) : n = 1, ...N\}$ extracted from a variable M_2 so that $\mathcal{L}(M_2) = N(0, w_2^n)$ and it is independent from M_1;
- We obtain N realization (y_1^n, y_2^n) of $\boldsymbol{Y}(1)$ by means of the relations $Y_1(1) = \mu_1 W_1 + \sigma_1 M_1$ and $Y_2(1) = \mu_2 W_2 + \sigma_2 M_2$.

For completeness we also simulate the marginal trajectories of the processes $\boldsymbol{G}(t)$ and $\boldsymbol{Y}(t)$ for the same values of the parameters considered in the scatter plots, thus for different levels of dependence. We summarize below the main steps to obtain the simulated process $\boldsymbol{Y}$. Notice that it is crucial the structure underlined in Remark 1. We simulate the value of the process $\boldsymbol{Y}$ at time points $\{n\Delta t, n = 0, ...N\}$ as follows

- We simulate N realizations $\{(x_1(n), x_2(n), z(n)), n = 1, ...N\}$ from the independent random variables $\mathcal{L}(X_1) = \Gamma((\frac{b}{\alpha} - a)\Delta t, \frac{b}{\alpha})$, $\mathcal{L}(X_2) = \Gamma((\frac{b}{\beta} - a)\Delta t, \frac{b}{\beta})$, $\mathcal{L}(Z) = \Gamma(a\Delta t, b)$;
- We obtain N realizations of the increments of $\boldsymbol{W}$, $\{(w_1(n), w_2(n)), n = 1, ..., N\}$ through the relations $W_1 = X_1 + Z$ and $W_2 = X_1 + Z$;
- We generate N independent random numbers $\{M_1(n) : n = 1, ...N\}$ extracted from a variable M_1 so that $\mathcal{L}(M_1) = N(0, 1)$ and N independent random numbers $\{M_2(n) : n = 1, ...N\}$ extracted from a variable M_2 so that $\mathcal{L}(M_2) = N(0, 1)$ and it is independent from M_1;
- The N independent simulated increment of the process $\boldsymbol{Y}$ are then $y_1(n) = \mu_1 \cdot w_1(n) + \sigma_1 M_1(n)\sqrt{w_1(n)}$ and $y_2(n) = \mu_2 \cdot w_2(n) + \sigma_2 M_2(n)\sqrt{w_2(n)}$;

- Finally for $i = 1, 2$ the simulated trajectories are:

$$Y_i(0) = 0,$$
$$Y_i(n\Delta t) = Y_i((n-1)\Delta t) + y_i(n).$$

We make all the simulation with the parameters $\mu_1 = -0.1$, $\mu_2 = -0.15$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$ fixed (recall that we also fix $b = 1$).

In the first figure we assume that the process $\boldsymbol{G}$ has identically distributed margins ($\alpha = \beta = 1$). We compare both the scatter plots and the trajectories of $\boldsymbol{G}$ and $\boldsymbol{Y}$ for different correlations, i.e. for $a = 1$, the limit case studied in [8], and for $a = 0.3$. In the second one we assume $\boldsymbol{G}$ has the margins with different distributions ($\alpha = 1.5$ and $\beta = 1$). We compare again the scatter plots and the trajectories for different correlations, i.e. for $a = 0.6$ and for $a = 0.3$.

Acknowledgments. *The Author is grateful to Professor Elisa Luciano for having inspired the paper and for her useful comments and suggestions.*

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